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Fidelity for States of Two Spin-$\frac{1}{2}$ Particles in Moving Frames

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Abstract Fidelity for the spin part of states of two spin-$\frac{1}{2}$ particles is investigated from the viewpoint of moving observers. Using a numerical approach, the behavior of the fidelity in terms of the boost parameter is described for different amounts of spin entanglement and momentum entanglement. It is shown that for the spin entangled states the fidelity decreases less than that of the case of spin product states and there are special cases for which the fidelity remains perfect regardless of moving observers' velocity. Generally, in the limit of boosts with speeds close to the speed of light, the fidelity saturates, i.e., it reaches to a constant value that depends on the amount of momentum entanglement and the width of the momentum distribution function.

Keywords Two-particle states · Wigner rotation · Reduced density matrix · Fidelity

1 Introduction

The role of special relativity in framing statements about quantum information is illustrated by the fact that quantum entanglement can depend on the reference frame of the observer. In practice, Lorentz transformations can change the entanglement of the spins of massive particles. One of the early works in this area has considered a single free spin-$\frac{1}{2}$ particle and by calculating the reduced density matrix, it is shown that the spin entropy is not a relativistic scalar [1].

The entanglement between the spins of a pair of particles may change because the spin and momentum become mixed when viewed by a moving observer [2]. Li an Du have investigated the quantum entanglement between the spins of spin-$\frac{1}{2}$ massive particles in moving frames, for the case that the momenta of the particles are entangled [3]. They have shown that, if the momenta of the pair are appropriately entangled, the entanglement between the spins of the Bell states remains maximal when viewed from any Lorentz-transformed frame.

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Recently, simple examples have been presented of Lorentz transformation that entangle the spins and momenta of two spin-$\frac{1}{2}$ particles with positive mass no sum of entanglements have been found to be unchanged.

Bell's inequality in moving frames has been considered in several papers [4–8]. The degree of violation of Bell's inequality will decrease with increasing the velocity of the observers if the directions of the measurements are fixed. However, this doesn't imply a breakdown of nonlocal correlation since the perfect anti-correlation is maintained in the appropriately chosen different directions.

In this article we are going to study the effect of Lorentz transformation on the fidelity for the spin part of states of two spin-$\frac{1}{2}$ particles. The concept of fidelity is a basic ingredient in communication theory and for any given communication scheme the fidelity is a quantitative measure of the accuracy of the transmission. In our argument the fidelity measures the similarity between spin parts of a state as observed in a boosted frame and the same state as observed in the laboratory frame. We argue on the spin product states and spin correlated states, separately, and for each case we consider the momentum part of state to be in two extreme cases of momentum product or perfectly correlated momentum.

2 Spin Fidelity under Lorentz Transformation

We investigate a bipartite state that in the momentum representation, as viewed from the rest frame of an observer, is denoted by

$$|\psi\rangle = \sum_{\sigma_1\sigma_2} \int\int d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, g_{\sigma_1\sigma_2}(\mathbf{p}_1, \mathbf{p}_2)\, |p_1, \sigma_1;\, p_2, \sigma_2\rangle, \tag{1}$$

where p_i is the four-momentum for the ith particle, and σ_i denotes the spin of the particle $(i = 1, 2)$. $g_{\sigma_1\sigma_2}(\mathbf{p}_1, \mathbf{p}_2)$ is the momentum and spin distribution function normalized as

$$\sum_{\sigma_1\sigma_2} \int\int d^3\mathbf{p}_1\, d^3\mathbf{p}_2\, |g_{\sigma_1\sigma_2}(\mathbf{p}_1, \mathbf{p}_2)|^2 = 1, \tag{2}$$

provided that $\langle p_1', \sigma_1';\, p_2', \sigma_2' | p_1, \sigma_1;\, p_2, \sigma_2\rangle = \delta^3(\mathbf{p}_1' - \mathbf{p}_1)\, \delta^3(\mathbf{p}_2' - \mathbf{p}_2)\, \delta_{\sigma_1'\sigma_1}\, \delta_{\sigma_2'\sigma_2}$.

For another observer in a Lorentz transformed frame, the state appears to be transformed by the unitary operator $U(\Lambda \otimes \Lambda)$. Therefore, the state as viewed by this observer is

$$\begin{aligned} |\tilde{\psi}\rangle = \sum_{\sigma_1\sigma_2\sigma_1'\sigma_2'} \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \sqrt{\frac{(\Lambda p_1)^0}{p_1^0}} \sqrt{\frac{(\Lambda p_2)^0}{p_2^0}}\, g_{\sigma_1\sigma_2}(\mathbf{p}_1, \mathbf{p}_2) \\ \times D_{\sigma_1'\sigma_1}(W(\Lambda, p_1))\, D_{\sigma_2'\sigma_2}(W(\Lambda, p_2))\, |\Lambda p_1, \sigma_1';\, \Lambda p_2, \sigma_2'\rangle, \end{aligned} \tag{3}$$

where $D_{\sigma_i'\sigma_i}(W(\Lambda, p_i))$ is the unitary representation of the Wigner rotation operator. Any Lorentz transformation can be written as a rotation followed by a boost. It turns out that tracing over the momentum after a rotation will not change the spin entanglement [9]. Therefore we can look only at pure boosts. Without loss of generality we may choose the boost in the z direction with a given velocity $V = \tanh\eta$.

In this section we are going to calculate the fidelity between the spin parts (called as spin fidelity) of the states $|\psi\rangle$ and $|\tilde{\psi}\rangle$ in some different cases of spin and momentum entanglement. To do this we need to the corresponding reduced density operators ϱ and $\tilde{\varrho}$ obtained by taking the trace over the momentum of $\rho = |\psi\rangle\langle\psi|$ and $\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}|$, that is

$$\varrho_{\sigma_1'\sigma_2',\sigma_1\sigma_2} = \int\int d^3\mathbf{p}_1\, d^3\mathbf{p}_2\; g_{\sigma_1'\sigma_2'}(\mathbf{p}_1, \mathbf{p}_2) g^*_{\sigma_1\sigma_2}(\mathbf{p}_1, \mathbf{p}_2), \tag{4}$$

and

$$\begin{aligned}\tilde{\varrho}_{\sigma_1'\sigma_2',\sigma_1\sigma_2} = &\sum_{\sigma_1''\sigma_2''}\sum_{\sigma_1'''\sigma_2'''}\int\int d^3\mathbf{p}_1\, d^3\mathbf{p}_2 \\ &\times \Big[D_{\sigma_1'\sigma_1''}(W(\Lambda_1, p_1)) g_{\sigma_1''\sigma_2''}(\mathbf{p}_1, \mathbf{p}_2) D_{\sigma_2'\sigma_2''}(W(\Lambda_2, p_2))\Big] \\ &\times \Big[D_{\sigma_1\sigma_1'''}(W(\Lambda_1, p_1)) g_{\sigma_1'\sigma_2'}(\mathbf{p}_1, \mathbf{p}_2) D_{\sigma_2\sigma_2'''}(W(\Lambda_2, p_2))\Big]^*,\end{aligned} \tag{5}$$

where the elements $D_{\sigma_i\sigma_i'''}(W(\Lambda_i, p_i)$ are given by (9). To calculate the spin fidelity we use the relation [10]

$$F_s = \big[\mathrm{Tr}\big(\sqrt{\varrho\,\tilde{\varrho}}\,\big)\big]^2 = \big(\sqrt{\nu_1} + \sqrt{\nu_2} + \sqrt{\nu_3} + \sqrt{\nu_4}\big)^2, \tag{6}$$

where ν_1, ν_2, ν_3 and ν_4 are the eigenvalues of $\varrho\tilde{\varrho}$.

In the rest frame of the observer, the four momentum of a particle moving with a velocity v, can be written as

$$p^\mu = [m\cosh\xi, m\sinh\xi(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)], \tag{7}$$

where $\tanh\xi = v$. In the following, we suppose that particles are moving along the x-axis, i.e., $\theta = \frac{\pi}{2}$ and $\phi = 0$, then the four momentum reduces to

$$p^\mu = (m\cosh\xi, m\sinh\xi, o, o). \tag{8}$$

This means that $\mathbf{p}$ has only one non-zero components $p^1 = m\sinh\xi$. Now, for the case of spin-$\frac{1}{2}$ particles with the same mass m, we have for each particle

$$D(W(\Lambda, p)) = \begin{bmatrix} \cos\frac{\Omega}{2} & -\sin\frac{\Omega}{2} \\ \sin\frac{\Omega}{2} & \cos\frac{\Omega}{2} \end{bmatrix}, \tag{9}$$

where

$$\cos\frac{\Omega}{2} = \frac{\cosh\frac{\eta}{2}\cosh\frac{\xi}{2}}{[\frac{1}{2} + \frac{1}{2}\cosh\eta\cosh\xi]^{\frac{1}{2}}}, \tag{10}$$

and

$$\sin\frac{\Omega}{2} = \frac{\sinh\frac{\eta}{2}\sinh\frac{\xi}{2}}{[\frac{1}{2} + \frac{1}{2}\cosh\eta\cosh\xi]^{\frac{1}{2}}}. \tag{11}$$

2.1 Spin Product States

These states are characterized by $\delta_{\sigma_1\uparrow}\delta_{\sigma_2\uparrow}$ as the spin part of the distribution function. We investigate these states in two extreme cases of momentum product and momentum correlated.

2.1.1 Momentum Product State

In this case we choose the momentum and spin distribution function as

$$g_{\sigma_1\sigma_2}(\mathbf{p}_1,\mathbf{p}_2)=f(\mathbf{p}_1)f(\mathbf{p}_2)\delta_{\sigma_1\uparrow}\delta_{\sigma_2\uparrow}, \tag{12}$$

where

$$f(\mathbf{p}_i)=\frac{1}{(\alpha^2\pi)^{\frac{1}{4}}}\sqrt{\delta(p_i^2)\delta(p_i^3)}\exp\left[-\frac{1}{2}\left(\frac{p_i^1}{\alpha}\right)^2\right], \tag{13}$$

where $p_i^1=m\sinh\xi_i$. Note that the parameter α determines the width of the momentum distribution function. Substituting (12) in (4) and (5), it turns out that the matrix $\varrho\tilde{\varrho}$ has only one non-zero eigenvalue $\tilde{\varrho}_{\uparrow\uparrow\uparrow\uparrow}$, that is

$$\begin{aligned}\tilde{\varrho}_{\uparrow\uparrow\uparrow\uparrow}&=\int d^3\mathbf{p}_1\int d^3\mathbf{p}_2|f(\mathbf{p}_1)|^2|f(\mathbf{p}_2)|^2|D_{\uparrow\uparrow}(\Omega_1)|^2|D_{\uparrow\uparrow}(\Omega_2)|^2\\&=\overline{\cos\frac{\Omega_1}{2}}\;\overline{\cos\frac{\Omega_2}{2}},\end{aligned} \tag{14}$$

where the overline is defined as $\overline{X}=\int d^3\mathbf{p}|f(\mathbf{p})|^2X(\mathbf{p})$. Then, assuming that particles move with the same velocity, that is $\xi_1=\xi_2=\xi$, the spin fidelity is obtained as

$$F_s(\eta)=\frac{4\gamma^2}{\pi}\left[\int_0^\infty d\xi\, e^{-\gamma^2\sinh^2\xi}\cosh\xi\frac{\cosh^2\frac{\eta}{2}\cosh^2\frac{\xi}{2}}{(\frac{1}{2}+\frac{1}{2}\cosh\eta\cosh\xi)}\right]^2, \tag{15}$$

where $\gamma=\frac{m}{\alpha}$ designates the ratio of mass of the particle to width of the momentum distribution function. There is no analytical solution for this integral, however some important result can be obtained by applying a numerical approach. Figure 1 shows a numerical sketch of F_s in terms of η for two values of $\gamma=1$ (the upper curve) and $\gamma=0.1$ (the lower curve).

2.1.2 Perfectly Correlated Momenta (Entangled Momenta)

In this case we choose

$$g_{\sigma_1\sigma_2}(\mathbf{p}_1,\mathbf{p}_2)=|f(\mathbf{p}_1)|\sqrt{\delta^3(\mathbf{p}_1-\mathbf{p}_2)}\,\delta_{\sigma_1\uparrow}\delta_{\sigma_2\uparrow}, \tag{16}$$

which as substituted in (4) and (5), leads to the following expression for the spin fidelity

$$F_s(\eta)=\frac{2\gamma}{\sqrt{\pi}}\int_0^\infty d\xi\, e^{-\gamma^2\sinh^2\xi}\cosh\xi\frac{\cosh^4\frac{\eta}{2}\cosh^4\frac{\xi}{2}}{(\frac{1}{2}+\frac{1}{2}\cosh\eta\cosh\xi)^2}. \tag{17}$$

Figure 2 shows a sketch of it for the values of $\gamma=1$ (the upper curve) and $\gamma=0.1$ (the lower curve). This figure is comparable with Fig. 1.

It is remarkable that as (15) and (17) imply, for the above discussed spin product states, as $\gamma\to 0$, the spin fidelity F_s approaches to zero. This means that wider the momentum distribution function, more decreasing the value of spin fidelity.

Fig. 1 The spin fidelity F_s versus the boost parameter η for a state comprised of a spin product part and a momentum product part. The *upper curve* is sketched for the ratio $\gamma = \frac{m}{\alpha} = 1$, while the *lower curve* is sketched for $\gamma = 0.1$. For a given mass, increasing the width parameter α leads to more decreasing of F_s. Asymptotically, the *upper curve* saturates to $F_s \cong 0.865$, while the *lower curve* reaches to $F_s \cong 0.427$

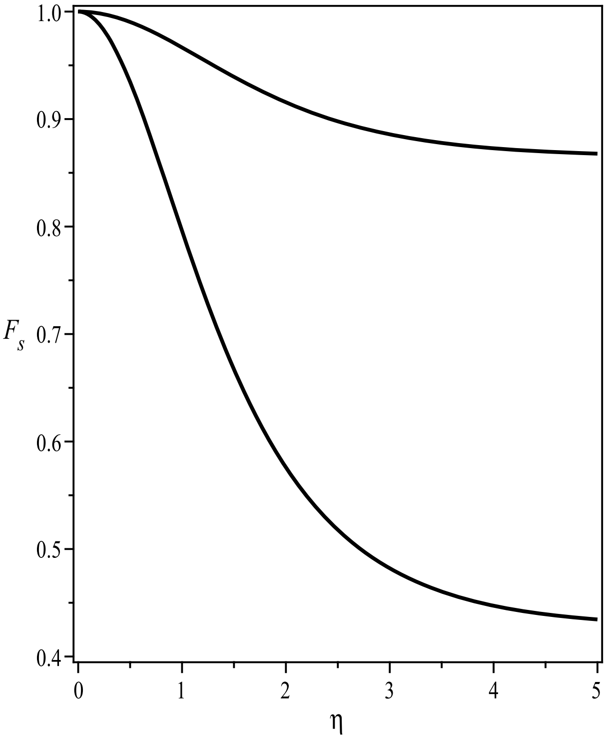
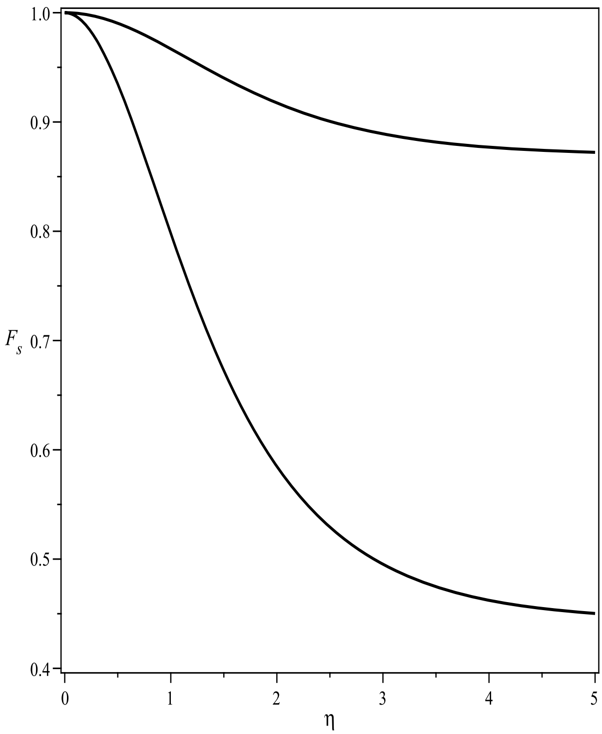


Fig. 2 F_s versus the boost parameter η for a state comprised of a spin product part and completely correlated momentum part. The *upper curve* is sketched for the ratio $\gamma = \frac{m}{\alpha} = 1$, while the *lower curve* is sketched for $\gamma = 0.1$. The behavior of these curves is very similar to that of curves of Fig. 1. However they don't really coincide. The *upper curve* asymptotically reaches to $F_s \cong 0.870$, while the *lower curve* nears to $F_s \cong 0.443$



2.2 Spin Singlet State

2.2.1 Momentum Product Case

In this case we choose

$$g_{\sigma_1\sigma_2}(\mathbf{p}_1,\mathbf{p}_2)=f(\mathbf{p}_1)f(\mathbf{p}_2)\frac{1}{\sqrt{2}}\left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow}-\delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right), \tag{18}$$

which as substituted in (4) and (5) leads to

$$\varrho=\frac{1}{2}\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix}, \tag{19}$$

and

$$\begin{aligned}\tilde{\varrho}_{\sigma_1'\sigma_2',\sigma_1\sigma_2}=\frac{1}{2}\int d^3p_1|f(p_1)|^2\int d^3p_2|f(p_2)|^2\\ \times[D_{\sigma_1'\uparrow}(\Omega_1)D_{\sigma_2'\downarrow}(\Omega_2)D_{\sigma_1\uparrow}(\Omega_1)D_{\sigma_2\downarrow}(\Omega_2)\\ -D_{\sigma_1'\downarrow}(\Omega_1)D_{\sigma_2'\uparrow}(\Omega_2)D_{\sigma_1\uparrow}(\Omega_1)D_{\sigma_2\downarrow}(\Omega_2)\\ -D_{\sigma_1'\uparrow}(\Omega_1)D_{\sigma_2'\downarrow}(\Omega_2)D_{\sigma_1\downarrow}(\Omega_1)D_{\sigma_2\uparrow}(\Omega_2)\\ +D_{\sigma_1'\downarrow}(\Omega_1)D_{\sigma_2'\uparrow}(\Omega_2)D_{\sigma_1\downarrow}(\Omega_1)D_{\sigma_2\uparrow}(\Omega_2)].\end{aligned} \tag{20}$$

Now constructing $\varrho\tilde{\varrho}$ and calculating its eigenvalues, we reach to the spin fidelity

$$F_s=\frac{1}{4}\left(\varrho^{(f)}_{\uparrow\downarrow\uparrow\downarrow}-\varrho^{(f)}_{\downarrow\uparrow\uparrow\downarrow}-\varrho^{(f)}_{\uparrow\downarrow\downarrow\uparrow}+\varrho^{(f)}_{\downarrow\uparrow\downarrow\uparrow}\right), \tag{21}$$

which as evaluated by (20) leads to

$$F_s=\frac{1}{2}\left(1+\overline{\cos\Omega_1}\;\overline{\cos\Omega_2}+\overline{\sin\Omega_1}\;\overline{\sin\Omega_2}\right). \tag{22}$$

Again we assume the same velocity for the particles and so F_s becomes

$$F_s=\frac{1}{2}\left(1+\overline{\cos\Omega}^2+\overline{\sin\Omega}^2\right). \tag{23}$$

It must be noted that $\overline{\sin\Omega}$ vanishes, then we obtain

$$F_s(\eta)=\frac{1}{2}+\frac{2\gamma^2}{\pi}\left[\int_0^\infty d\xi\, e^{-\gamma^2\sinh^2\xi}\cosh\xi\left(\frac{\cosh\eta+\cosh\xi}{1+\cosh\eta\cosh\xi}\right)\right]^2. \tag{24}$$

Now using the numerical approach we solve the integral. Figure 3 shows a sketch of this F_s.

2.2.2 Perfectly Correlated Momenta

We choose in this case

$$g_{\sigma_1\sigma_2}(\mathbf{p}_1,\mathbf{p}_2)=|f(p_1)|\sqrt{\delta^3(\mathbf{p}_1-\mathbf{p}_2)}\frac{1}{\sqrt{2}}\left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow}-\delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right), \tag{25}$$

which as substituted in (4) and (5), interestingly leads to a perfect fidelity, that is $F_s=1$.

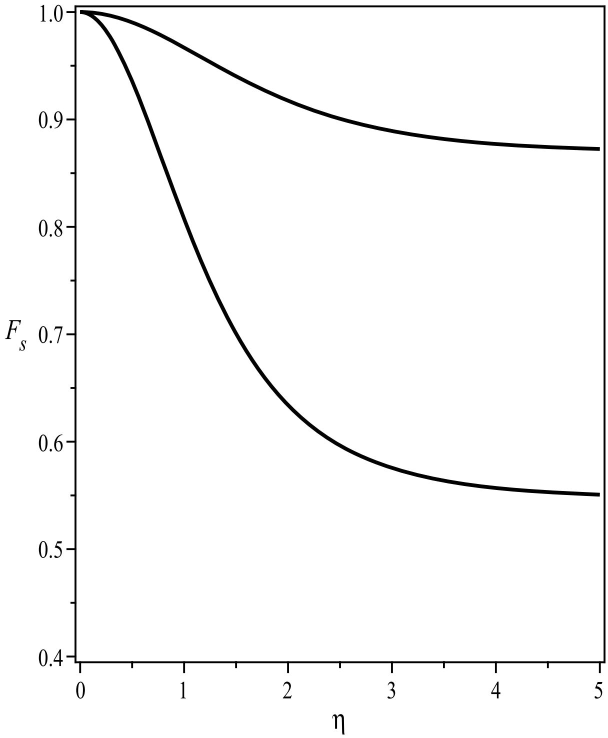


Fig. 3 F_s versus the boost parameter η for a state comprised of a spin singlet (or triplet) part and a momentum product part. The *upper curve* is sketched for the ratio $\gamma = \frac{m}{\alpha} = 1$, while the *lower curve* is sketched for $\gamma = 0.1$. In these cases F_s never becomes less than 0.5, although $\gamma \to 0$

2.3 Spin Triplet State

2.3.1 Momentum Product State

We have

$$g_{\sigma_1\sigma_2}(\mathbf{p}_1, \mathbf{p}_2) = f(\mathbf{p}_1) f(\mathbf{p}_2) \frac{1}{\sqrt{2}} \left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right), \tag{26}$$

which leads to

$$F_s = \frac{1}{2}\left(1 + \overline{\cos\Omega}^2 - \overline{\sin\Omega}^2\right), \tag{27}$$

for the case of particles with the same velocity. Recall that $\overline{\sin\Omega} = 0$, then we reach to the same expression (24) for the spin fidelity. So the curves sketched in Fig. 3 describe F_s in this case, as well.

It must be noted that as (23) an (27) indicate, the spin fidelity for the singlet and the triplet states with momentum product part never becomes less than $\frac{1}{2}$, even if $\gamma \to 0$.

2.3.2 Perfectly Correlated Momenta

$$g_{\sigma_1\sigma_2}(\mathbf{p}_1, \mathbf{p}_2) = |f(\mathbf{p}_1)|^2 \sqrt{\delta^3(\mathbf{p}_1 - \mathbf{p}_2)} \frac{1}{\sqrt{2}} \left(\delta_{\sigma_1\uparrow}\delta_{\sigma_2\downarrow} + \delta_{\sigma_1\downarrow}\delta_{\sigma_2\uparrow}\right). \tag{28}$$

This again leads to a perfect fidelity $F_s = 1$ as the singlet case.

3 Conclusion

We conclude that the behavior of the spin fidelity as a function of boost parameter, depends mostly on the amount of spin entanglement. Moreover, it depends on the amount of momentum entanglement and the width of momentum distribution function. Generally, more decreasing in the spin fidelity occurs for the spin product states in comparison with the spin correlated states. In the spin product cases, a little difference exists between the behavior of the fidelity for momentum product state and perfectly correlated momentum state. For both states the fidelity falls by increasing the boost parameter, can even reach to a zero value. However, in the spin correlated cases, i.e., singlet and triplet cases, a significant difference exists between the behavior of the fidelity for momentum product state and perfectly correlated momentum state; for the momentum product state the fidelity falls from unity, though never becomes less than $\frac{1}{2}$, while for the perfectly correlated momentum state the fidelity remains perfect, i.e., equals the unity, regardless of the boost velocity.

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